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A non-linear theory of turbulence onset in a shear flow

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Turbulence onset is considered in viscous incompressible flows. Development of fluctuations in an infinite flow with a constant gradient of mean velocity $\partial U_1/\partial x_2 = \text{const.}$, $U_2 = U_3 = 0$ is rigorously treated.

It is shown that two-dimensional eddy fluctuations, with infinitesimal initial amplitude A and scale of initial eddies l, increase in this flow so that the maximum ratio $\max \epsilon(t)/\epsilon(0)$ of their energy at the moment t to the initial energy exceeds any prescribed value as the Reynolds number $R = (\partial U_1/\partial x_2)\, l^2/\nu$ increases. The analysis of the non-linear equations obtained in the paper which describe development of fluctuations with a finite amplitude leads to the conclusion that there exists a 'stability barrier' $\tilde{A}(R)$ for the initial amplitude of eddy fluctuations. If $A < \tilde{A}(R)$, then fluctuations decay as $t \to \infty$, and if $A > \tilde{A}(R)$ the energy of fluctuations does not decay. As $R \to \infty$, $\tilde{A}(R) \to 0$ according to the inequalities

$$l(\partial U_1/\partial x_2)\, K_2/R^{\frac{2}{3}} \leqslant \tilde{A}(R) \leqslant l(\partial U_1/\partial x_2)\, K_1/R^{\frac{2}{3}}.$$

It is shown that the non-linear mechanism of preventing turbulence from decay involves generation of large-scale turbulent oscillations which then transmit energy to small-scale motions.

The described mechanism of turbulence onset from small eddies in shear flows appears to be of universal character. It is interesting that several qualitative characteristics of turbulence observed in various shear flows can be rigorously deduced even in a model where disturbances remain two-dimensional.

Nomenclature

$\mathbf{U}(\mathbf{x}, t)$	average flow velocity;
$\mathbf{u}(\mathbf{x}, t)$	fluctuation velocity;
p	pressure;
ρ	density;
ν	kinematic viscosity;
$b_{ij}(\mathbf{x}, \mathbf{x}', t)$	correlation function of fluctuations;
$\phi_{ij}(\mathbf{k}, t)$	spectral fluctuation tensor;
$\mathbf{k}$	wave-number vector;
$R = (\partial U_1/\partial x_2)\, l^2/\nu$	local Reynolds number;
l	scale of initial vortices;
A	amplitude;
$\beta = \partial U_1/\partial x_2$	mean shear.

1. Development of disturbances with an infinitesimal initial amplitude

Onset, development and maintenance of turbulence in a viscous incompressible fluid are caused by the energy of an averaged flow and external random effects. Investigation of the mechanism of onset and development of turbulence demands therefore consideration of non-uniform turbulent flows.

The Navier–Stokes equations of motion may be written in the form

$$\left.\begin{aligned}\frac{\partial}{\partial t}(U_i+u_i)+(U_j+u_j)\frac{\partial(U_i+u_i)}{\partial x_j} &= \nu\nabla^2(U_i+u_i)-\frac{1}{\rho}\frac{\partial p}{\partial x_i},\\ \frac{\partial(U_i+u_i)}{\partial x_i} &= 0.\end{aligned}\right\} \tag{1}$$

Here $\mathbf{U}(\mathbf{x},t) = \overline{\mathbf{v}(\mathbf{x},t)}$ is the averaged value of the true velocity

$$\mathbf{v}(\mathbf{x},t) = \mathbf{U}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t);$$

the bar denotes mathematical expectation; summation is carried out over repeated subscripts.

The velocity of averaged motion $\mathbf{U}(\mathbf{x},t)$ and correlation function of turbulent fluctuations $b_{ij}(\mathbf{x},\mathbf{x}',t) = \overline{u_i(\mathbf{x},t)\,u_j(\mathbf{x}',t)}$ are main properties of the flow.

If $\mathbf{U} = 0$, the energy of turbulent fluctuations decreases. Turbulence develops, on the other hand, in regions where the velocity gradient of the average motion is not zero.

If, when averaging, terms of the order of $\overline{u_i\,u_j\,u_k}$ are neglected, it follows from (1) that

$$\begin{aligned}\frac{\partial b_{ij}}{\partial t}+\left[U_k(\mathbf{x},t)\frac{\partial}{\partial x_k}+U_k(\mathbf{x}',t)\frac{\partial}{\partial x'_k}\right]b_{ij}(\mathbf{x},\mathbf{x}',t)&\\ +\frac{\partial U_i}{\partial x_k}b_{kj}+\frac{\partial U_j}{\partial x'_k}b_{ik} = -\frac{1}{\rho}\left(\frac{\partial b_{pj}}{\partial x_i}+\frac{\partial b_{ip}}{\partial x'_j}\right)&+\nu(\nabla^2_x b_{ij}+\nabla^2_{x'}b_{ij}),\end{aligned} \tag{2}$$

$$\frac{\partial}{\partial x_i}b_{ij}(\mathbf{x},\mathbf{x}',t) = \frac{\partial}{\partial x'_j}b_{ij}(\mathbf{x},\mathbf{x}',t) = 0. \tag{3}$$

Here, functions $b_{pi} = \overline{pu_i}$ are eliminated by the condition of incompressibility.

Equation (2) corresponds to a linear approximation in (1) for the function $\mathbf{u}(\mathbf{x},t)$. Later, in § 2, a non-linear theory describing development of fluctuations with a finite amplitude is described.

We turn now to the case when the flow is non-uniform but initial disturbances are such that the maximum turbulence scale l is less than the characteristic scale of the averaged flow L. Fluctuations $\mathbf{u}(\mathbf{x},t)$ are rapidly changing functions of $\mathbf{x},t$; averaged velocities $\mathbf{U}(\mathbf{x},t)$ change slowly. Therefore the function

$$b_{ij}(\mathbf{x},\mathbf{x}',t) = d_{ij}(\mathbf{x}-\mathbf{x}',\tfrac{1}{2}(\mathbf{x}+\mathbf{x}'),t)$$

changes rapidly depending on $\mathbf{x}-\mathbf{x}'$ and slowly depending on $\frac{1}{2}(\mathbf{x}+\mathbf{x}')$; when $|\mathbf{x}-\mathbf{x}'| \to \infty$ the function $b_{ij}(\mathbf{x},\mathbf{x}',t)$ quickly approaches zero. In this case it may be considered that the tensor $b_{ij}(\mathbf{x},\mathbf{x}',t)$ depends on the vector $\mathbf{x}-\mathbf{x}'$ and the tensor $\partial U_i/\partial x_k$ alone.

Actually, transforming (2) to new variables $\mathbf{r} = \mathbf{x}-\mathbf{x}'$ and $\mathbf{y} = \frac{1}{2}(\mathbf{x}+\mathbf{x}')$ and retaining only the first derivatives with respect to the co-ordinates of the slowly changing functions $\mathbf{U}(\mathbf{x},t)$, we obtain

$$\left(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial y_k}\right)d_{ij}(\mathbf{r},\mathbf{y},t)+\frac{\partial U_k}{\partial y_l}r_l\frac{\partial}{\partial r_k}d_{ij}(\mathbf{r},\mathbf{y},t)+\frac{\partial U_i}{\partial y_k}d_{kj}+\frac{\partial U_j}{\partial y_k}d_{ik} = \frac{1}{\rho}\left(\frac{\partial d_{pj}}{\partial r_i}-\frac{\partial d_{ip}}{\partial r_j}\right)+2\nu\frac{\partial^2 d_{ij}}{\partial r_k^2}. \quad (4)$$

We shall introduce the spectral tensor

$$\phi_{ij}(\mathbf{k},\mathbf{x},t) = \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}}\,\overline{u_i(\mathbf{x}+\tfrac{1}{2}\mathbf{r})\,u_j(\mathbf{x}-\tfrac{1}{2}\mathbf{r})}\,d\mathbf{r}, \quad (5)$$

which depends on the co-ordinates $\mathbf{x}$ not explicitly, but through slowly varying functions $\mathbf{U}(\mathbf{x},t)$ (Hasen 1962).

Then it follows from (2) that

$$\left(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial x_k}\right)\phi_{ij}(\mathbf{k},\mathbf{x},t)-\frac{\partial U_k}{\partial x_l}k_k\frac{\partial}{\partial k_l}\phi_{ij}+2\nu k^2\phi_{ij} +\frac{\partial U_s}{\partial x_k}\left(\delta_{si}-2\frac{k_i k_s}{k^2}\right)\phi_{kj}+\frac{\partial U_s}{\partial x_k}\left(\delta_{sj}-2\frac{k_j k_s}{k^2}\right)\phi_{ik} = 0. \quad (6)$$

We shall consider the plane parallel flow $\mathbf{U}(\mathbf{x},t)$ where

$$U_1 = U_1(x_2), \quad \partial U_1/\partial x_2 = \beta \neq 0, \quad U_2 = U_3 = 0$$

(the axis x_1 is directed along the flow, and the axis x_2 across it). In this case equations (6) are simplified. The solution is of the form

$$\phi_{22}(\mathbf{k},t) = \phi_{22}^0(k_1, k_2+k_1\beta t, k_3)\left[\frac{k_1^2+(k_2+k_1\beta t)^2+k_3^2}{k_1^2+k_2^2+k_3^2}\right]^2 \times\exp\{-2\nu[k^2t+k_1k_2\beta t^2+\tfrac{1}{3}k_1^2\beta^2t^3]\}. \quad (7)$$

For a plane flow we have

$$\phi_{12} = -(k_2/k_1)\,\phi_{22}, \quad \phi_{12} = \phi_{21}, \quad \phi_{11} = (k_2/k_1)^2\,\phi_{22}. \quad (8)$$

Here, ϕ_{ij}^0 are values of the functions at $t = 0$. When $U = 0$, which corresponds to homogeneous and isotropic turbulence, and $\phi_{ij}^0 = (\delta_{ij}k^2-k_ik_j)$, the solution (7)–(8) becomes the well-known expression of Millionshchikov (1939, 1941). The fluctuation energy is

$$\epsilon(t) = \int_{-\infty}^{\infty}\phi_{ii}(\mathbf{k},t)\,d\mathbf{k} = \epsilon_1(t)+\epsilon_2(t)+\epsilon_3(t). \quad (9)$$

For an initial wavy disturbance which is periodic in the direction of the flow we have

$$\phi_{ij}^0(\mathbf{k}) = A_{ij}[\delta(k_1-a)+\delta(k_1+a)]\,\delta(k_2)\,\delta(k_3). \quad (10)$$

In this case, (7) and (8) show that the energy of the disturbance decreases monotonically as t grows. Thus in a flow with a constant shear, such an initial wavy disturbance decreases monotonically. For the more general initial disturbance

$$\phi_{ij}^0(\mathbf{k}) = A(\delta_{ij}k^2-k_ik_j)\,[\delta(\mathbf{k}-\mathbf{k}_0)+\delta(\mathbf{k}+\mathbf{k}_0)], \quad (11)$$

where $\mathbf{k}_0 = (k_1, k_2)$, we obtain from (6) when $\nu = 0$:

$$\epsilon_2(k_1, k_2, t) = A k_1^2 \left(\frac{k_1^2 + k_2^2}{k_1^2 + (k_2 - k_1 \beta t)^2} \right)^2; \tag{12}$$

$$\epsilon_1(k_1, k_2, t) = A \left(\frac{(k_1^2 + k_2^2)(k_2 - k_1 \beta t)}{k_1^2 + (k_2 - k_1 \beta t)^2} \right)^2. \tag{13}$$

However, equations (7) and (8) show that there exist initial disturbances for which fluctuation energy starts to grow with t and increases in a finite time interval for a fixed local Reynolds number $R = \beta l^2/\nu$; moreover, as R increases, the maximum ratio $\max \epsilon(t)/\epsilon(0)$ becomes larger than any prescribed value.

In fact for eddy disturbances consisting of eddies of the type

$$u_1(\mathbf{x}) = -\frac{\partial}{\partial x_2} e^{-\frac{1}{2}x^2\alpha^2}, \quad u_2(\mathbf{x}) = \frac{\partial}{\partial x_1} e^{-\frac{1}{2}x^2\alpha^2}, \tag{14}$$

located at random in the plane (x_1, x_2), we have

$$\phi_{ij}^0(\mathbf{k}) = A(\delta_{ij} k^2 - k_i k_j) e^{-k^2/\alpha^2}. \tag{15}$$

From (7), (9), (15), using dimensionless variables $t^* = \beta t$, $k^* = kl = k/\alpha$ (it is natural to take $1/\alpha$ as the scale of initial turbulent disturbances l), we obtain (Hasen 1962)

$$\begin{aligned} \frac{\epsilon(t^*)}{\epsilon(0)} > \frac{\epsilon_2(t^*)}{\epsilon(0)} &= \int_{-\infty}^{\infty} \frac{k_1^{*2}}{2\pi} \exp\{-[k_1^{*2} + (k_2^* + k_1^* t^*)^2]\} \\ &\times \left[\frac{k_1^{*2} + (k_2^* + k_1^* t^*)^2 + k_3^{*2}}{k_1^{*2} + k_2^{*2} + k_3^{*2}} \right]^2 \exp\left\{ -\frac{2}{R} (t^* k^{*2} + t^{*2} k_1^* k_2^* + \tfrac{1}{3} t^{*3} k_1^{*2}) \right\} d\mathbf{k}^* \\ &> \int_{Q(t^*)} \frac{k_1^{*2}}{6\pi} \left[\frac{k_1^{*2} + (k_2^* + k_1^* t^*)^2 + k_3^{*2}}{k_1^{*2} + k_2^{*2} + k_3^{*2}} \right]^2 \\ &\times \exp\{-(2/R)(t^* k^{*2} + t^{*2} k_1^* k_2^* + \tfrac{1}{3} t^{*3} k_1^{*2})\} d\mathbf{k}^*, \end{aligned} \tag{16}$$

where $Q(t^*)$ is the region prescribed by the inequalities

$$|k_2^* + k_1^* t^*| \leqslant \tfrac{1}{3}, \quad |k_1^*| \leqslant \tfrac{1}{3}, \quad |k_3^*| \leqslant \tfrac{1}{3}.$$

Let M be any prescribed value. When $t^* \leqslant N(M)$ and $R > N^3(M)$, we have for all k^* in $Q(t^*)$ the inequality

$$\exp\{-(2/R)(t^* k^{*2} + t^{*2} k_1^* k_2^* + \tfrac{1}{3} t^{*3} k_1^{*2})\} > \tfrac{1}{3}.$$

Hence

$$\frac{\epsilon(t^*)}{\epsilon(0)} > \int_{Q(t^*)} \frac{k_1^{*2}}{18\pi} \frac{[k_1^{*2} + (k_2^* + k_1^* t^*)^2 + k_3^{*2}]^2}{(k_1^{*2} + k_2^{*2} + k_3^{*2})^2} d\mathbf{k}^*. \tag{17}$$

Evaluating integral (17) first with respect to k_2^* and then with respect to k_1^* (with k_3^* replaced in the numerator by zero, and by unity in the denominator), we find

$$\frac{\epsilon(t^*)}{\epsilon(0)} > \frac{t^{*4}}{3^9(1 + \frac{1}{2} t^{*2})} \tag{18}$$

when $t^* \leqslant N(M)$, $R > N^3(M)$. Having chosen $N(M) = 2\sqrt{(3^9 M)}$, we find that at $\sqrt{(3^9 M)} < t^* < 2\sqrt{(3^9 M)}$ and $R > 8(3^9 M)^{\frac{3}{2}}$, the relation $\epsilon(t^*)/\epsilon(0) > M$ is fulfilled.

Thus, no matter how large M is, a Reynolds number $R = \tilde{R}(M)$ may be always found such that when $\sqrt{(3^9 M)} < t < 2\sqrt{(3^9 M)}$ the energy of infinitesimal initial eddies has increased more than M times. Here, however, the amplitude of finite, although initially small, fluctuations may be out of the region of linear approximation for $\mathbf{u}(\mathbf{x}, t)$ and (7)–(8) cease to be valid. In order to trace further development of fluctuations, it is necessary to analyse non-linear equations for $\mathbf{u}(\mathbf{x}, t)$. The results of such an analysis, which is made in the subsequent section, show that the energy of turbulent fluctuations does not decay if the maximum fluctuation energy at the initial stage of turbulence development is out of the region of linear approximation. However, if the initial amplitude of fluctuations is sufficiently small, so that the highest possible amplitude is within the region of linear approximation, then fluctuations decay when $t \to \infty$, energy exchange and generation of fluctuations of different scales being absent.

Thus the shear flow is stable with respect to sufficiently small eddies, but the height of 'stability barrier' for the initial amplitude of eddies approaches zero as R grows.

2. A non-linear theory of turbulence onset

We shall consider the further development of turbulent fluctuations in a flow with a uniform profile of the mean flow $\partial U_1/\partial x_2 = \beta$, in the case of a finite value of the initial amplitude.

We shall apply the Fourier transformation to the fluctuation velocity

$$\tilde{\mathbf{u}}(\mathbf{k}, t) = \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{u}(\mathbf{x}, t)\, d\mathbf{x}. \tag{19}$$

Eliminating the pressure from the Navier–Stokes equations by means of the incompressibility condition, we obtain the following equation for $\tilde{\mathbf{u}}(\mathbf{k}, t)$:

$$\frac{\partial \tilde{u}_i(\mathbf{k}, t)}{\partial t} - \beta k_1 \frac{\partial \tilde{u}_i}{\partial k_2} + \beta \delta_{i1} \tilde{u}_2 + \nu k^2 \tilde{u}_i - 2\beta \frac{k_1 k_i}{k^2} \tilde{u}_2 = i\left(\frac{k_i k_l}{k^2} - \delta_{il}\right) \int_{-\infty}^{\infty} k'_j\, \tilde{u}_l(\mathbf{k}', t)\, \tilde{u}_j(\mathbf{k} - \mathbf{k}', t)\, d\mathbf{k}'. \tag{20}$$

We shall consider a two-dimensional flow, $\tilde{\mathbf{u}} = (\tilde{u}_1(k_1, k_2, t), \tilde{u}_2(k_1, k_2, t))$. In this case from the compressibility condition $k_1 \tilde{u}_1 + k_2 \tilde{u}_2 = 0$ we find, for some ψ,

$$\tilde{u}_1 = -k_2 \psi(k_1, k_2, t), \quad \tilde{u}_2 = k_1 \psi(k_1, k_2, t). \tag{21}$$

Then from (20) and (21), the non-linear equation for $\psi(k_1, k_2, t)$ follows:

$$\frac{\partial \psi(\mathbf{k}, t)}{\partial t} - \beta k_1 \frac{\partial \psi}{\partial k_2} + \nu k^2 \psi - 2\beta \frac{k_1 k_2}{k^2} \psi = \frac{i}{k^2} \int_{-\infty}^{\infty} (k_2 k'_1 - k_1 k'_2)(k_2 k'_2 + k_1 k'_1)\, \psi(\mathbf{k}', t)\, \psi(\mathbf{k} - \mathbf{k}', t)\, d\mathbf{k}'. \tag{22}$$

We shall consider the solution of (22) with an initial condition

$$\psi(\mathbf{k}, 0) = (A/\alpha^2) \exp(-k^2/\alpha^2), \tag{23}$$

that corresponds to initial vortices. We assume $\psi(\mathbf{k},t) = D(\mathbf{k},t)+iB(\mathbf{k},t)$, introduce the dimensionless Reynolds number $R = \beta/\alpha^2\nu$ and transfer to dimensionless variables $k^* = k/\alpha$; $t^* = \beta t$; $A^* = A\alpha/\beta$. When the value of A^* is fixed, the Reynolds number may increase due to decrease of the viscosity. The asterisk in t^* and k^* will be omitted in the subsequent portion.

We shall solve equation (22) with the initial condition (23) by the method of successive approximations. As the first (linear) approximation we have

$$D_1(\mathbf{k},t) = A^*\left(\frac{k_1^2+(k_2+k_1t)^2}{k_1^2+k_2^2}\right)^2 \exp\left[-k_1^2-(k_2+k_1t)^2\right] \times \exp\{-(1/R)(k^2t+k_1k_2t^2+\tfrac{1}{3}k_1^2t^3)\}, \quad B_1(\mathbf{k},t) = 0. \quad (24)$$

We denote $D_1(\mathbf{k},t) = A^*G_1(\mathbf{k},t)$. The solution is expressed as a series in powers of A^*:

$$\psi(\mathbf{k},t) = \sum_{n=1}^{\infty}[D_n(\mathbf{k},t)+iB_n(\mathbf{k},t)], \quad (25)$$

where the functions $D_n(\mathbf{k},t)$, $B_n(\mathbf{k},t)$ for $n \geqslant 2$ are the solutions of the following equations:

$$\frac{\partial D_n(\mathbf{k},t)}{\partial t} - k_1\frac{\partial D_n}{\partial k_2} + \frac{k^2D_n}{R} - 2\frac{k_1k_2}{k^2}D_n = -\frac{1}{k^2}\int_{-\infty}^{\infty}(k_2k_1'-k_1k_2')(k_2k_2'+k_1k_1') \times \sum_{p+q=n}[D_p(\mathbf{k}',t)B_q(\mathbf{k}-\mathbf{k}',t)+D_p(\mathbf{k}-\mathbf{k}',t)B_q(\mathbf{k}',t)]\,d\mathbf{k}', \quad (26)$$

$$\frac{\partial B_n(\mathbf{k},t)}{\partial t} - k_1\frac{\partial B_n}{\partial k_2} + \frac{k^2B_n}{R} - 2\frac{k_1k_2}{k^2}B_n = \frac{1}{k^2}\int_{-\infty}^{\infty}(k_2k_1'-k_1k_2')(k_2k_2'+k_1k_1') \times \sum_{p+q=n}[D_p(\mathbf{k}',t)D_q(\mathbf{k}-\mathbf{k}',t)-B_p(\mathbf{k}',t)B_q(\mathbf{k}-\mathbf{k}',t)]\,d\mathbf{k}', \quad (27)$$

with the initial conditions

$$D_n(\mathbf{k},0) = B_n(\mathbf{k},0) = 0 \quad (n \geqslant 2). \quad (28)$$

We have seen that $B_1(\mathbf{k},t) = 0$ and $D_1(\mathbf{k},t)$ is defined by formula (24); equations (26) and (27) give $D_{2m}(\mathbf{k},t) = 0$, $B_{2m+1}(\mathbf{k},t) = 0$. We shall estimate $|D_{2m+1}(\mathbf{k},t)|$, $|B_{2m}(\mathbf{k},t)|$. The following lemmas are basic for the estimation of these functions.

Lemma 1. Consider the integral

$$y_1(k_1,k_2,t) = \frac{1}{k^2}\int_{-\infty}^{\infty}(k_2k_1'-k_1k_2')(k_2k_2'+k_1k_1')\,G_1(\mathbf{k}',t)\,G_1(\mathbf{k}-\mathbf{k}',t)\,d\mathbf{k}'. \quad (29)$$

When $t \geqslant R^{\frac{1}{3}}$ const., $y_1(k_1,k_2,t)$ is estimated as

$$|y_1(k_1,k_2,t)| \sim \exp\{-\tfrac{1}{2}[k_1^2+(k_2+k_1t)^2]-(1/2R)[k^2t+k_1k_2t^2+k_1^2t^3/3]\} \times\left[\frac{k_1^2+(k_2+k_1t)^2}{k_1^2+k_2^2}\right]^2\frac{R^{\epsilon}}{k^2t^{\gamma}}|k_1k_2+\text{terms of lower order with respect to } t,\ k_2|, \quad (30)$$

where $\epsilon = \frac{1}{2}, \gamma = \frac{3}{2}$ for $t \sim R^{\frac{1}{3}}$ and $\epsilon = 2, \gamma = 3$ for $t \sim R$. Similarly for the integrals

$$y_n(k_1,k_2,t) = \frac{1}{k^2}\int_{-\infty}^{\infty}(k_2k_2'-k_1k_2')(k_2k_2'+k_1k_1')\,G_l(\mathbf{k}',t)\,G_{n-l}(\mathbf{k}-\mathbf{k}',t)\,d\mathbf{k}', \quad (31)$$

where l is an integer with $1 \leqslant l \leqslant n$ and

$$G_m(\mathbf{k}, t) = \left[\frac{k_1^2+(k_2+k_1t)^2}{k_1^2+k_2^2}\right]^m \times \exp\left\{-\frac{1}{m}[k_1^2+(k_2+k_1t)^2]-\frac{1}{mR}(k^2t+k_1k_2t^2+k_1^2t^3/3)\right\},$$

we estimate

$$|y_n(k_1,k_2,t)| \sim \exp\left\{-\frac{1}{n}[k_1^2+(k_2+k_1t)^2]-\frac{1}{nR}(k^2t+k_1k_2t^2+k_1^2t^3/3)\right\} \times\left[\frac{k_1^2+(k_2+k_1t)^2}{k_1^2+k_2^2}\right]^n \frac{R^\epsilon}{k^2t^\gamma}|k_1k_2+\text{terms of lower power with respect to } t,\, k_2|, \tag{32}$$

where $\epsilon = \frac{1}{2}$, $\gamma = \frac{3}{2}$ for $t \sim R^{\frac{1}{3}}$ and $\epsilon = 2$, $\gamma = 3$ for $t \sim R$.

The proof of lemma 1 is based on the fact that as $t \to \infty$

$$\left.\begin{aligned}\exp\{-2(k_1'-\tfrac{1}{2}k_1)^2t^3/3R\} &\Rightarrow \sqrt{(3\pi R/2t^3)}\,\delta(k_1'-\tfrac{1}{2}k_1);\\ \exp\{-2(k_2'-\tfrac{1}{2}k_2)^2t/R\} &\Rightarrow \sqrt{(\pi R/2t)}\,\delta(k_2'-\tfrac{1}{2}k_2).\end{aligned}\right\} \tag{33}$$

The integrals (31), (29) may be 'cut', i.e. estimated by integrals over the region

$$|k_1'-\tfrac{1}{2}k_1| \leqslant 3\sqrt{(3R/2t^3)}; \quad |k_2'-\tfrac{1}{2}k_2| \leqslant \min(3;\, 3\sqrt{(R/2t)}). \tag{34}$$

As t increases, this region becomes narrower with respect to k_1' more rapidly than with respect to k_2'. Estimation of integrals (31), (29) over the region (34) (for $t \geqslant R^{\frac{1}{3}}$ const.) first with respect to k_1' and then to k_2' (where the function $G_l(\mathbf{k}'')\,G_{n-l}(\mathbf{k}-\mathbf{k}')$ near the point $k_1' = \frac{1}{2}k_1$, $k_2' = \frac{1}{2}k_2$ is expanded in a Taylor series) yields the estimates (32), (30). The integral over the region external to (34) gives the terms of lower order with respect to t, k_2 which are referred to in expressions (32), (30).

Lemma 2. Consider the solution of the non-uniform linear equation

$$\frac{\partial B(\mathbf{k},t)}{\partial t}-k_1\frac{\partial B}{\partial k_2}+\frac{k^2}{R}B-2\frac{k_1k_2}{k^2}B = \frac{k_1k_2}{k^2}G_n(k_1,k_2,t)\frac{1}{1+t^p} \quad (p>0); \tag{35}$$

with the initial condition $B(\mathbf{k};\, 0) = 0$.

Then the following estimate for the solution is valid for $t\in[0,t_0]$ ($t_0 \sim R^{\frac{1}{3}}$):

$$\frac{C_1}{n(1+t^p)}G_n(\mathbf{k},t) \leqslant |B(\mathbf{k},t)| \leqslant \frac{C_2}{n(1+t^p)}G_n(\mathbf{k},t),$$

where $C_1, C_2 = \text{const.}$

Lemma 2 is verified directly by integration of (35).

Lemma 3. Signs of the terms of higher order with respect to t in the functions $D_{2m+1}(\mathbf{k},t)$, $B_{2m}(\mathbf{k},t)$ alter in the following way:

$$D_{2m+1} \sim (-1)^m; \quad B_{2m} \sim (-1)^m \quad (m=0,1,2,3,\ldots). \tag{36}$$

Because of this fact, all these terms enter into the sum in the right-hand side of equations (26), (27) with the same signs.

Lemma 3 is proved by the method of mathematical induction.

From lemmas 1, 2, 3, we deduce lemma 4.

Lemma 4. For the terms of highest order with respect to t in the functions $B_{2m}(\mathbf{k},t)$, $D_{2m+1}(\mathbf{k},t)$, the following estimations hold when $t \geqslant R^{\frac{1}{3}}$ const.:

$$|D_{2m+1}(\mathbf{k},t)| \sim A^*\left(\frac{A^*R^\epsilon}{t^\gamma K}\right)^{2m} G_{2m+1}(\mathbf{k},t); \tag{37}$$

$$|B_{2m}(\mathbf{k},t)| \sim A^*\left(\frac{A^*R^\epsilon}{t^\gamma K}\right)^{2m-1} G_{2m}(\mathbf{k},t); \tag{38}$$

where $\epsilon = \frac{1}{2}, \gamma = \frac{3}{2}$ for $t \sim R^{\frac{1}{3}}$ and $\epsilon = 2, \gamma = 3$ for $t \sim R$; $K = \text{const}$. When $t \sim R^{\frac{1}{3}}$, we have

$$|D_{2m+1}(\mathbf{k},t)| = G_{2m+1}(\mathbf{k},t)\, A^*(A^*R^{\frac{1}{2}})^{2m}\left|\frac{\text{const}}{(t^{\frac{3}{2}}K)^{2m}} + \text{terms of lower power with respect to } t\right|.$$

Now consider $\int_{-\infty}^{\infty} \psi(\mathbf{k},t)\,d\mathbf{k}$ that is connected with the energy of turbulent disturbances. Then for $t \sim R^{\frac{1}{3}}$

$$\left|\int_{-\infty}^{\infty} D_{2m+1}(\mathbf{k},t)\,d\mathbf{k}\right| \sim A^*\left(A^*\frac{R^{\frac{1}{2}}}{t^{\frac{3}{2}}K}\right)^{2m} t^{4m}; \tag{39}$$

$$\left|\int_{-\infty}^{\infty} B_{2m}(\mathbf{k},t)\,d\mathbf{k}\right| \sim A^*\left(A^*\frac{R^{\frac{1}{2}}}{t^{\frac{3}{2}}K}\right)^{2m-1} t^{4m-2}. \tag{40}$$

Deduction of these estimations is the same as for energy $\epsilon(t)$ in the linear theory of § 1 of this paper.

From (39) and (40) it is evident that for $t \sim R^{\frac{1}{3}}$, the energy of disturbances remains finite and tends to zero as t increases, if $A^* \leqslant K_2/R^{\frac{2}{3}}$, and becomes infinite for finite time $t \sim R^{\frac{1}{3}}$ if $A^* \geqslant K_1/R^{\frac{2}{3}}$ (where $K_1, K_2 = \text{const.}$).

From the above we may conclude that there exists a 'stability barrier' $\tilde{A}^*(R)$, where

$$K_2/R^{\frac{2}{3}} \leqslant \tilde{A}^*(R) \leqslant K_1/R^{\frac{2}{3}} \tag{41}$$

for the initial amplitude A of turbulent fluctuations. It should be such a quantity that if $A < (\beta/\alpha)\tilde{A}^*(R)$, fluctuations decay as $t \to \infty$, and if $A > (\beta/\alpha)\tilde{A}^*(R)$, fluctuations (in a free stream with a constant mean shear) grow and their energy does not decrease. As $R \to \infty$ the 'stability barrier $\tilde{A}^*(R)$ becomes zero.

We shall analyse the non-linear mechanism which prevents turbulence from decay. If the initial conditions are given in the form

$$\psi(\mathbf{k},0) = \delta(\mathbf{k}-\mathbf{k}_0) + \delta(\mathbf{k}+\mathbf{k}_0), \tag{42}$$

corresponding to a plane wave with the wave vector $\mathbf{k}_0$, then the solution of (22) will include all waves which are integral multiples of $\mathbf{k}_0$. If

$$\psi(\mathbf{k},0) = \delta(\mathbf{k}-\mathbf{k}_1) + \delta(\mathbf{k}+\mathbf{k}_1) + \delta(\mathbf{k}-\mathbf{k}_2) + \delta(\mathbf{k}+\mathbf{k}_2), \tag{43}$$

corresponding to two plane waves with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, then all waves arise with the wave vectors $n\mathbf{k}_1$; $m\mathbf{k}_2$; $n\mathbf{k}_1 - m\mathbf{k}_2$; $n, m = \pm 1, \pm 2, \ldots$. In this case fluctuations of a larger scale, in addition to fluctuations of a smaller scale, appear in the course of time. The former take their energy from the gradient of the mean flow and transmit it to the fluctuations of small scale that are generated by them. If $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel but not commensurable, then waves are excited whose wave vectors form a dense set.

When we consider the interaction of a finite number of waves, the problem is the solution of a system of ordinary differential equations of the first order.

Both the solutions of (22) and those of the set of equations for spectral tensors of turbulent fluctuations, including the higher moments (Hasen 1962, 1963), are similar.

The present mechanism of turbulence onset from small eddies in shear flows with $\beta \neq 0$ is universal, since in an incompressible fluid any velocity field contains an eddy component and in the regions where turbulence develops the shear is non-zero.

The conclusions agree with the reported experimental data on the turbulence onset: they give an explanation of the phenomenon of delay in transition to a turbulent flow, i.e. of the increase of the Reynolds number for transition to a turbulent flow when the level of disturbances to a laminar flow decreases. Turbulent 'spots' and the intermittent character of turbulence onset may be explained by the fact that energy of random small initial turbulent fluctuations at a certain Reynolds number is large enough not at all points simultaneously, but in some places located at random. In these places, 'turbulent spots' appear and (as agrees well with the present theory) when the Reynolds number grows or the level of disturbances in the flow increases, the sizes and number of such spots rise. Since the initial level of disturbance to the flow cannot be infinitesimal, infinite delay of transition to a turbulent flow is impossible.

Admittedly the real situation is still more complicated than that analysed in this paper, since the disturbances are three-dimensional. It is interesting, however, that the above qualitative conclusions can be deduced even if purely two-dimensional disturbances are assumed.

REFERENCES

Hasen, E. M. 1962 On the theory of inhomogeneous turbulence. *Dokl. Akad. Nauk SSSR* **147**, 60–63.

Hasen, E. M. 1963 On the non-linear theory of turbulence. *Dokl. Akad. Nauk SSSR* **153**, 1284–1287.

Millionshchikov, M. D. 1939 Degeneration of homogeneous isotropic turbulence in a viscous incompressible fluid. *Dokl. Akad. Nauk SSSR*, **2**, 236–240.

Millionshchikov, M. D. 1941 To the theory of homogeneous isotropic turbulence. *Izv. Akad. Nauk. SSSR, Ser. Geografich. i Geofiz.* nos. 4–5, 433–446.